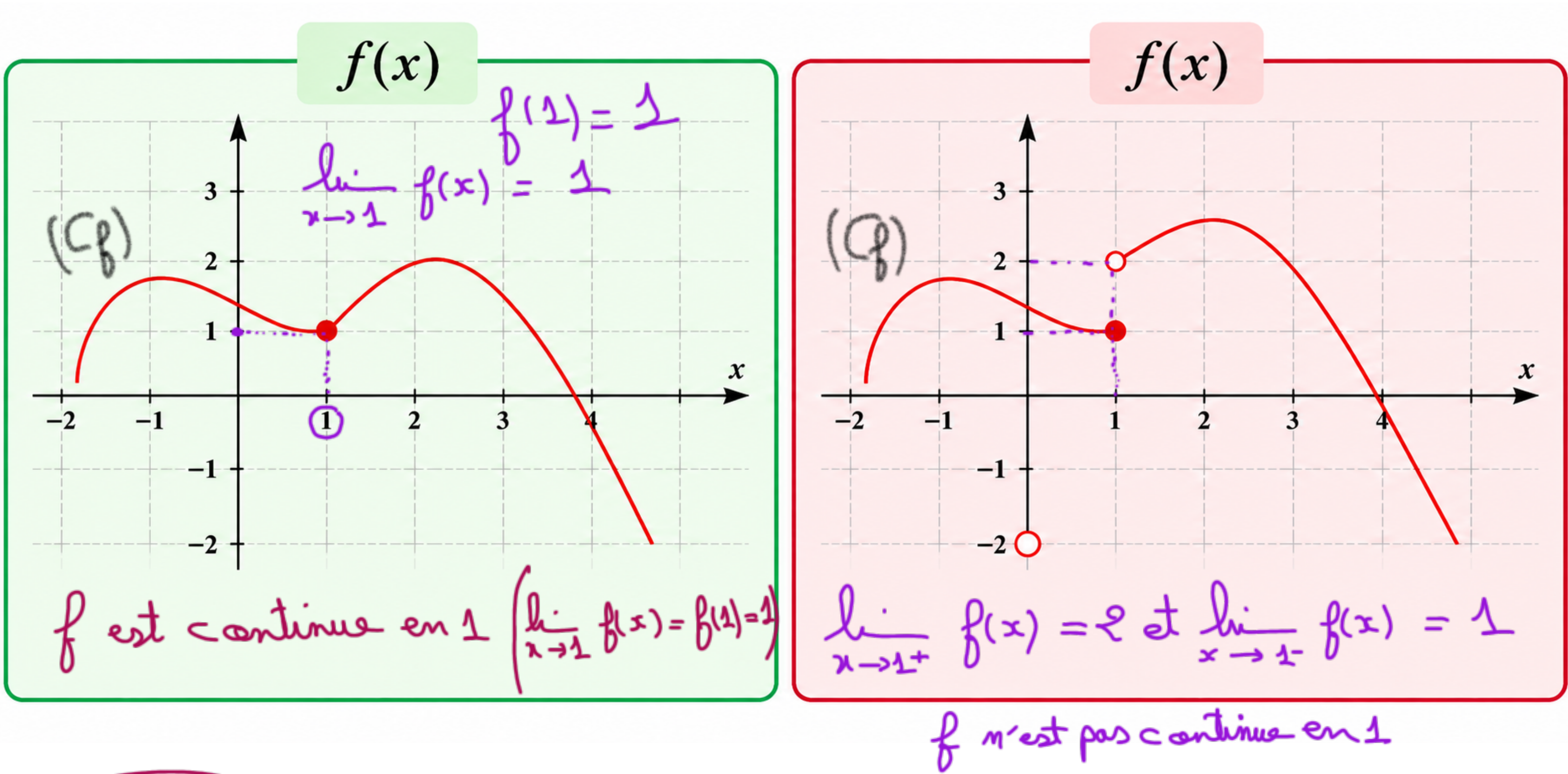


SÉRIE D'EXERCICES

Étude de la Continuité en un point

Activité : Notion graphique de la continuité

Observez les deux représentations graphiques ci-dessous :



Définition à retenir :

f est continue en a :

Si $\lim_{x \rightarrow a} f(x) = f(a)$

Exercice 1 : Étude directe de la continuité

Dans chacun des cas suivants, étudier la continuité de la fonction f au point x_0 :

$$1) \begin{cases} f(x) = \frac{\sqrt{2x+2}-1}{4x^2-1} & \text{si } x \neq -\frac{1}{2} \\ f\left(-\frac{1}{2}\right) = -\frac{1}{4} \end{cases}$$

$$\text{et } x_0 = -\frac{1}{2}$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$4x^2 - 1 = (2x)^2 - 1^2$$

$$\text{on a } \lim_{x \rightarrow -\frac{1}{2}} f(x) = \lim_{x \rightarrow -\frac{1}{2}} \frac{\sqrt{2x+2}-1}{4x^2-1} = \lim_{x \rightarrow -\frac{1}{2}} \frac{(\sqrt{2x+2}-1)(\sqrt{2x+2}+1)}{(4x^2-1)(\sqrt{2x+2}+1)}$$

$$= \lim_{x \rightarrow -\frac{1}{2}} \frac{(\sqrt{2x+2})^2 - (1)^2}{((2x)^2 - (1)^2)(\sqrt{2x+2}+1)} = \lim_{x \rightarrow -\frac{1}{2}} \frac{2x+2-1}{(2x-1)(2x+1)(\sqrt{2x+2}+1)}$$

$$= \lim_{x \rightarrow -\frac{1}{2}} \frac{\cancel{2x+1}}{(2x-1)\cancel{(2x+1)}(\sqrt{2x+2}+1)} = \lim_{x \rightarrow -\frac{1}{2}} \frac{1}{(2x-1)(\sqrt{2x+2}+1)}$$

$$= \frac{1}{(2 \times -\frac{1}{2} - 1)(\sqrt{2 \times -\frac{1}{2} + 2} + 1)} = \frac{1}{-2 \times 2} = -\frac{1}{4} = f\left(-\frac{1}{2}\right)$$

Puisque $\lim_{x \rightarrow -\frac{1}{2}} f(x) = f\left(-\frac{1}{2}\right)$; alors f est continue en $-\frac{1}{2}$

$$2) \begin{cases} f(x) = \frac{(1-\cos x)^2}{\sin^3 x} & \text{si } x \neq 0 \\ f(0) = 0 \end{cases}$$

$$\text{et } x_0 = 0$$

$$\text{on a } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{(1-\cos(x))^2}{\sin^3(x)}$$

$$= \lim_{x \rightarrow 0} (1-\cos(x))^2 \times \frac{1}{\sin^3(x)}$$

$$= \lim_{x \rightarrow 0} \frac{(1-\cos(x))^2}{(x^2)^2} \times \frac{1}{(x^3)^3}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1-\cos(x)}{x^2} \right)^2 \times \frac{x^3}{\sin^3(x)} \times x$$

$$= \left(\frac{1}{2}\right)^2 \times 1 \times 0 = 0 = f(0)$$

donc f est continue en 0.

Rappel

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \text{ et } \lim_{x \rightarrow 0} \frac{x}{\sin(x)} = 1$$

$$\lim_{x \rightarrow 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

$$3) \begin{cases} f(x) = \frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3} & \text{si } x \neq 2 \\ f(2) = \frac{8}{9} \end{cases}$$

et $x_0 = 2$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{(x - \sqrt{x+2})(x + \sqrt{x+2})(\sqrt{4x+1} + 3)}{(\sqrt{4x+1} - 3)(\sqrt{4x+1} + 3)(x + \sqrt{x+2})}$$

$$= \lim_{x \rightarrow 2} \frac{(x^2 - (x+2))(\sqrt{4x+1} + 3)}{(4x+1 - 9)(x + \sqrt{x+2})} = \lim_{x \rightarrow 2} \frac{(x^2 - x - 2)(\sqrt{4x+1} + 3)}{(4x - 8)(x + \sqrt{x+2})}$$

$$\begin{array}{r|l} x^2 - x - 2 & x - 2 \\ \hline x^2 - 2x & \downarrow \\ \hline 0 \quad x - 2 & \\ \hline 0 \quad x - 2 & \\ \hline 0 & \end{array}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x+1)(\sqrt{4x+1} + 3)}{4(x-2)(x + \sqrt{x+2})}$$

$$= \lim_{x \rightarrow 2} \frac{(x+1)(\sqrt{4x+1} + 3)}{4(x + \sqrt{x+2})} = \frac{(2+1)(\sqrt{4 \cdot 2 + 1} + 3)}{4(2 + \sqrt{2+2})} = \frac{18}{16}$$

$$= \frac{18}{16} = \frac{9}{8} \neq f(2) \quad \text{donc } f \text{ n'est pas continue en } 2$$

$$\left(\lim_{x \rightarrow 2} f(x) \neq f(2) \right)$$

$$4) \begin{cases} f(x) = \frac{2x^3 - x^2 + 4x - 20}{x^3 - 8} & \text{si } x \neq 2 \\ f(2) = 2 \end{cases}$$

et $x_0 = 2$

$$x^3 - 8 = x^3 - 2^3$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \quad \triangle !$$

$$\lim_{x \rightarrow 2} \frac{2x^3 - x^2 + 4x - 20}{x^3 - 8} = \lim_{x \rightarrow 2} \frac{(x-2)(2x^2 + 3x + 10)}{(x-2)(x^2 + 2x + 4)} = \lim_{x \rightarrow 2} \frac{2x^2 + 3x + 10}{x^2 + 2x + 4} = \frac{24}{12} = 2 = f(2)$$

donc f est continue en 2.

$$\begin{array}{r|l} 2x^3 - x^2 + 4x - 20 & x - 2 \\ \hline -2x^3 - 4x^2 & \downarrow \\ \hline 0 \quad 3x^2 + 4x & \\ \hline -3x^2 - 6x & \downarrow \\ \hline 0 \quad 10x - 20 & \\ \hline -10x - 20 & \downarrow \\ \hline 0 & \end{array} \quad \begin{array}{l} 2x^2 + 3x + 10 \\ \hline \end{array}$$

$$5) \begin{cases} f(x) = \frac{\sqrt{x^2+1} - \sqrt{2}}{x^2+3x+2} & \text{si } x \neq -1 \\ f(-1) = \frac{\sqrt{2}}{2} \end{cases}$$

et $x_0 = -1$

$$\lim_{x \rightarrow -1} \frac{\sqrt{x^2+1} - \sqrt{2}}{x^2+3x+2} = \lim_{x \rightarrow -1} \frac{(\sqrt{x^2+1} - \sqrt{2})(\sqrt{x^2+1} + \sqrt{2})}{(x^2+3x+2)(\sqrt{x^2+1} + \sqrt{2})}$$

$$= \lim_{x \rightarrow -1} \frac{x^2+1-2}{(x+1)(x+2)(\sqrt{x^2+1} + \sqrt{2})}$$

$$\begin{array}{r|l} x^2+3x+2 & x+2 \\ \hline x^2+x & \downarrow \\ \hline 0 & 2x+2 \\ & -2x+2 \\ \hline & 00 \end{array}$$

$$= \lim_{x \rightarrow -1} \frac{x^2-1}{(x+1)(x+2)(\sqrt{x^2+1} + \sqrt{2})}$$

$$= \lim_{x \rightarrow -1} \frac{(x-1)(\cancel{x+1})}{(\cancel{x+1})(x+2)(\sqrt{x^2+1} + \sqrt{2})} = \lim_{x \rightarrow -1} \frac{x-1}{(x+2)(\sqrt{x^2+1} + \sqrt{2})}$$

$$= \frac{-2}{\sqrt{2} + \sqrt{2}} = \frac{-2 \times \sqrt{2}}{2\sqrt{2} \times \sqrt{2}} = \frac{-2\sqrt{2}}{4} = -\frac{\sqrt{2}}{2} \neq f(-1)$$

donc f n'est pas continue en -1

$$6) \begin{cases} f(x) = \frac{2 \cos^2 x - \cos x - 1}{x^2} & \text{si } x \neq 0 \\ f(0) = -\frac{3}{2} \end{cases} \quad \text{et } x_0 = 0$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2 \cos^2(x) - \cos(x) - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \cos^2(x) - \cos(x) - 1 - 2 + 1}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos^2(x) - 2 - \cos(x) + 1}{x^2} = \lim_{x \rightarrow 0} \frac{2(\cos^2(x) - 1) + 1 - \cos(x)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{2(\cos^2(x) - 1)}{x^2} + \frac{1 - \cos(x)}{x^2} = \lim_{x \rightarrow 0} \frac{2(\cos(x) - 1)(\cos(x) + 1)}{x^2} + \frac{1 - \cos(x)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{-2(1 - \cos(x))(\cos(x) + 1)}{x^2} + \frac{1 - \cos(x)}{x^2} \\ &= \lim_{x \rightarrow 0} -2 \left(\frac{1 - \cos(x)}{x^2} \right) (\cos(x) + 1) + \frac{1 - \cos(x)}{x^2} = -2 \times \frac{1}{2} (1 + 1) + \frac{1}{2} \\ &= -2 + \frac{1}{2} = -\frac{3}{2} = f(0) ; \text{ donc } f \text{ est continue en } 0 \end{aligned}$$

Exercice 2 : Détermination d'un paramètre

Dans chacun des cas suivants, déterminer la valeur du réel a pour que la fonction f soit continue en x_0 :

$$1) \begin{cases} f(x) = \frac{\sqrt{x} - 1}{x^2 - 3x + 2} & \text{si } x \neq 1 \\ f(1) = a \end{cases} \quad \text{et } x_0 = 1$$

Puisque f est continue en 1 $\Leftrightarrow \lim_{x \rightarrow 1} f(x) = f(1) = a$

$$\begin{aligned} \text{ona } \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x^2 - 3x + 2)(\sqrt{x} + 1)} &= \lim_{x \rightarrow 1} \frac{x - 1}{(x - 1)(x - 2)(\sqrt{x} + 1)} \\ \lim_{x \rightarrow 1} \frac{1}{(x - 2)(\sqrt{x} + 1)} &= \frac{1}{(1 - 2)(\sqrt{1} + 1)} = -\frac{1}{2} \end{aligned}$$

$$\text{d'où : } a = -\frac{1}{2}$$

$$\mathbf{2)} \quad \begin{cases} f(x) = \frac{\sin(\pi x)}{x-1} & \text{si } x \neq 1 \\ f(1) = a \end{cases} \quad \text{et} \quad x_0 = 1$$

$$\mathbf{3)} \quad \begin{cases} f(x) = \frac{\tan x - \sqrt{3}}{x - \frac{\pi}{3}} & \text{si } x \neq \frac{\pi}{3} \\ f\left(\frac{\pi}{3}\right) = a \end{cases} \quad \text{et} \quad x_0 = \frac{\pi}{3}$$

$$4) \quad \begin{cases} f(x) = \frac{ax^2 + (a+1)x + 1}{2x^2 + x - 1} & \text{si } x \neq -1 \\ f(-1) = \frac{1}{3} \end{cases} \quad \text{et } x_0 = -1$$

"L'excellence n'est pas un acte, mais une habitude."